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The stability of an inverted pendulum when there are rapid random oscillations of the suspension point☆

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Abstract

The stability of the upper equilibrium position of a pendulum when the suspension point makes rapid random oscillations of small amplitude, is investigated. A class of random oscillations that make the system stable with unit probability for small friction is indicated. It is shown that, if there is no friction, there is no stability, which, as is well known, is not the case for harmonic oscillations of the suspension point. Some general results concerning the impossibility of stochastic stabilization of Hamiltonian systems are proved.

The stability of the upper equilibrium position of a pendulum when the suspension point makes rapid harmonic oscillations is an interesting phenomenon, which has been known for a long time, at least since 1908 (see Ref. 1). It was included by the Nobel prize-winner P. L. Kapitza in a general physically rigorous theory of motion in a rapidly oscillating external field.[2] Moreover, Kapitza proved the stability of the pendulum by direct physical experiment. To put Kapitsa's theory on a sound mathematical basis requires, in particular, the Kolmogorov–Arnol'd–Moser theory of Hamiltonian systems, close to integrable ones (see Ref. 3). The problem is simplified considerable for a non-ideal pendulum which dissipates energy, and a solution of the problem was obtained by Bogolyubov[4] before the work of Kapitza.

The topic of this paper is a stochastic analog of Kapitsa's problem and is close to that of Refs. 5 and 6, where, in particular, some other types of random perturbations of the motion of a pendulum are considered. We refer to Refs. 7 and 5 for general results on the stabilization of an unstable linear system using random noise, which, in turn, can be regarded as a stochastic analog of the main result obtained in Ref. 8, where such stabilization using fast harmonic oscillations was constructed. The methods used here are, however, different from those in Refs. 5–7, being more general and simple.

At the physical level of rigour the possibility of stabilizing the upper vertical position of a pendulum by means of rapid random oscillations of the suspension was established in Ref. 9. More general problems of the stochastic stabilization of conservative systems were studied in Ref. 10. The results obtained are not, however, directly related either to the problem of the stability of the upper equilibrium position of a pendulum with small friction, or to the problem of its instability when there is no friction.

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1. Basic equations

The equation of motion of an inverted pendulum of length l with a vertically vibrating suspension point for small angles ϕ with the vertical has the form

$$\ddot{\phi} = \left(\frac{\ddot{\zeta}}{l} + \frac{g}{l}\right)\phi - \nu\dot{\phi},$$

where $\zeta = \zeta(t)$ is the height of the suspension point and ν is the coefficient of friction. The point $(\phi, \dot{\phi}) = (0, 0)$ corresponds to the upper equilibrium position. We assume that the motion of the suspension point is fast and of small amplitude. More accurately, $\zeta(t) = az(\omega t)$, where the amplitude a is small compared to l and the frequency ω is large compared to the natural frequency $\omega_0 = \sqrt{g/l}$ of the pendulum. Direct verification shows that this dynamic system can be described by the equations

$$x' = \varepsilon y, \quad y' = -\varepsilon^2\delta y + \varepsilon(z'' + \varepsilon^2 k^2)x, \tag{1.1}$$

where

$$x = \phi, \quad y = \dot{\phi}/(\omega\varepsilon), \quad k^2 = g/(l\omega^2\varepsilon^4), \quad \delta = \nu/(\omega\varepsilon^2), \quad \varepsilon^2 = a/l.$$

Here, the prime denotes differentiation with respect to the fast time $\tau = \omega t$, and the motion of suspension point is described by a random process $az(\omega t)$. We will assume the parameter ε to be small, whilst the parameters k and δ are of the order of unity. In other words, $\omega \sim \varepsilon^{-2}\omega_0$.

2. Stationary random processes with fairly strong mixing

We will assume that the function z, which describes the motion of the suspension point, is the trajectory of a stationary random process with the properly of fairly strong mixing. In this section we formulate the mixing condition precisely and prove a lemma on integration with respect to time of processes with strong mixing. In the next section, using these ideas, we will state and prove a theorem on the stability of the upper equilibrium position of a pendulum, the suspension point of which makes rapid random oscillations of small amplitude.

Suppose we are given a stationary random process $f(t)$ and a continuous and increasing family of σ-algebras $\mathcal{F}_s$ containing the σ-algebra $\sigma(f(\tau), \tau \le s)$, generated by values of the process up to the instant s. In what follows all stationary processes are assumed to have a finite mathematical expectation.

We will say that such a process has fairly strong mixing if $\mathbf{E}|f(t)|^2 < \infty$ and

$$\|\mathbf{E}(f(s) - (\mathbf{E}f(s))|\mathcal{F}_t)\| \le c(s-t)\|f(t) - (\mathbf{E}f(t))\|; \quad \int_0^{+\infty} c(s)ds < \infty,$$

where $\mathbf{E}$ is the mathematical expectation, $s \ge t$ and $\|\cdot\|$ stands for the L_2-norm.

It turns out that, apart from small error, any process with sufficiently strong mixing and zero mean can be integrated, and the integral is again a stationary process. The precise statement is as follows.

Lemma 1. *Let $t \mapsto f(t)$ be a stationary process with zero mean and fairly strong mixing. Then, a stationary square integrable process F exists such that $F(t) = \int_0^t f(s)\,ds + M(t)$, where M is a square integrable martingale with stationary increments.*

Proof. We introduce the following notation

$$I(\alpha, \beta, t) = \mathbf{E}\left(\int_\alpha^\beta f(s)ds \middle| \mathcal{F}_t\right)$$

and write down the *formal* identity

$$I(0, \infty, t) = \int_0^t f(s)ds + I(t, \infty, t).$$

Note that on a formal level the process $t \mapsto I(t, \infty, t)$ is stationary, while the process $t \mapsto I(0, \infty, t)$ is a martingale. Formally speaking, we must put

$$F(t) = -I(t, \infty, t), \quad M(t) = -I(0, \infty, t).$$

To justify these formal calculations we put

$$F_T(t) = -I(t, T', t), \quad M_T(t) = -I(0, T', t); \quad T' = \max(T, t).$$

Then the strong mixing property immediately gives that the processes $F_T(t)$, $M_T(t)$ converge in L_2 as $T \to \infty$ and the limit processes have the required properties.□

We will now present a typical example of a stationary random process with strong mixing which can arise with regard to the pendulum problem. Let z be a unique stationary solution with the zero mean of the following Ito equations

$$dz = z'dt, \quad dz' = -z'dt - zdt + dw, \tag{2.1}$$

where w is a standard Wiener process. It can be seen that $\mathbf{E}z'^2 = 1/2$. Then the process $t \mapsto z'(t)^2 - 1/2$ is a stationary process with the strong mixing and zero mean. Moreover, the quantities

$$F(t) = -(z^2(t) + z'^2(t))/2, \quad M(t) = -\int_0^t z'(s)dw(s)$$

give the decomposition of Lemma 1.

3. Stability theorem

Theorem 1. *Suppose the process z from (1.1) is continuously differentiable and stationary with the zero mean,* $\exp(\varepsilon^2|z|)$ *has finite mathematical expectation if* ε *is sufficiently small, and the process* z'^2 *has fairly strong mixing. Then, the null solution of system (1.1) is exponentially stable with unit probability, if* ε *is sufficiently small and* $k^2 < \mathbf{E}z'^2$.

Proof. The theorem is proved by making several (invertible, linear and simplectic) changes of variables, which reduce system (1.1) to a form, where we can apply the Lyapunov function technique. The first two changes of variables are

$$x = x_1, \quad y = y_1 + \varepsilon z' x_1, \tag{3.1}$$

$$x_1 = x_2, \quad y_1 = y_2 - \varepsilon^3 \delta z x_2, \tag{3.2}$$

which transform system (1.1) to the form

$$\begin{aligned} x_2' &= \varepsilon y_2 + \varepsilon^2 z' x_2 - O(\varepsilon^4) x_2 \\ y_2' &= -\varepsilon^2 \delta y_2 - \varepsilon^2 z' y_2 + \varepsilon^3 (k^2 - z'^2) x_2 + O(\varepsilon^4) y_2 + O(\varepsilon^4) x_2. \end{aligned} \tag{3.3}$$

We can avoid the first step of the method of averaging (transformation (3.1)) by using canonically conjugate variables instead of x, y. Namely, we note that the pair

$$x_1 = \phi, \quad y_1 = [\dot\phi - \dot\zeta l \phi] = m^{-1} \partial L / \partial \dot\phi$$

is just the pair of canonically conjugate variables (apart from a factor m^{-1}) for the quadratic Hamiltonian corresponding to a mathematical pendulum.

From now on notation like $O(\varepsilon^4)$ is used for quantities such that their absolute value is less than $\varepsilon^4\, \xi(\tau)$, where ξ is a certain positive *stationary* process such that the mathematical expectation $\mathbf{E}|\,\xi(\tau)|$ is uniformly bounded with respect to ε. Now we apply the transformation

$$x_2 = \exp(\varepsilon^2 z) x_3, \quad y_2 = \exp(-\varepsilon^2 z) y_3, \tag{3.4}$$

which nullifies terms $O(\varepsilon^2)$ in the preceding system. Namely,

$$\begin{aligned} x_3' &= \varepsilon y_3 + O(\varepsilon^4)x_3 + O(\varepsilon^3)y_3 \\ y_3' &= -\varepsilon^2\delta y_3 + \varepsilon^3(k^2 - z'^2)x_3 + O(\varepsilon^4)y_3 + O(\varepsilon^4)x_3. \end{aligned} \tag{3.5}$$

Next we would like to apply the transformation

$$x_3 = x_4, \quad y_3 = y_4 + \varepsilon^3 S(\tau)x_4, \tag{3.6}$$

where $dS = (-z'^2 + \mathbf{E}z'^2)d\tau$, to simplify the term $\varepsilon^3(k^2 - z'^2)x_3$. Unfortunately, this is impossible if we require that S must be a *stationary* process. However, the decomposition of Lemma 1 about strong mixing processes enables us to find a square integrable stationary process S such that $dS = (-z'^2 + \mathbf{E}z'^2)d\tau + dM$, where M is a square integrable martingale. Now, if we use this process S in transformation (3.6), the system takes the form

$$\begin{aligned} dx &= (\varepsilon y + O(\varepsilon^4)x + O(\varepsilon^3)y)d\tau \\ dy &= (-\varepsilon^2\delta y + \varepsilon^3(k^2 - \langle z'^2\rangle)x + O(\varepsilon^4)y + O(\varepsilon^4)x)d\tau - \varepsilon^3 x dM. \end{aligned} \tag{3.7}$$

where the subscript 4 is omitted.

It now remains to show that the latter system is (with unit probability) exponentially stable, for all the preceding transformations increase (with respect to the fast time τ) more slowly than $\exp(\mu\tau)$, where μ is arbitrarily small. To do this we use the usual Lyapunov functions technique.[11] Namely, we take the quadratic Lyapunov function

$$V(x, y) = (ax^2 + 2bxy + cy^2)/2$$
$$a = \varepsilon^2(\delta^2/2 + A^2), \quad b = \varepsilon\delta/2, \quad c = 1, \quad A^2 = \mathbf{E}z'^2 - k^2 = \text{const} > 0$$

and calculate the differential of the process $V(x_\tau, y_\tau)$. We obtain

$$dV \le -\varepsilon^2 fVd\tau + O(\varepsilon^3)Vd\tau + BdM,$$

where f is a positive constant (which depends on δ and A, but not on ε), B is a quadratic form in x, y such that $B = O(V)$ (here, the constant in O is independent of ε and is, in fact, a *stationary* process) and $O(\varepsilon^3)$ has the same meaning as in (3.3). Here, the inequality for the differentials is just a formal expression of inequality for the corresponding integrals.

Now, Ito's formula shows that

$$d\ln V \le -\varepsilon^2 fd\tau + O(\varepsilon^3)d\tau - b^2 d\langle M, M\rangle/2 + bdM, \quad b = B/V,$$

where $\langle M, M\rangle$ stands for the quadratic variation[12] of the martingale M. We have

$$\ln V(\tau) \le \ln V(0) - \varepsilon^2 f\tau + \varepsilon^3\int_0^\tau \xi(s)ds + \int_0^\tau b(s)dM(s), \tag{3.8}$$

where ξ and b are some stationary processes (functions of z, z') with finite mathematical expectation. In view of the almost certain convergence of the quantities $\tau^{-1}\int_0^\tau \xi(s)ds$ as $\tau \to \infty$ to an integrable random quantity (by the ergodic theorem), we arrive at the conclusion that the first integral in (3.8) is negligible compared to $\varepsilon^2 f\tau$ as $\varepsilon \to 0$ and τ is large. The latter integral is, in fact, a Wiener process in the new time scale

$$\theta(\tau) = \int_0^\tau b(s)^2 d\langle M, M\rangle(s).$$

Thus, it does not exceed $O(\sqrt{\theta \log\log\theta})$ (by the iterated logarithm law), for large t (in fact, the bound θ^λ, where $1/2 < \lambda < 1$ is quite sufficient for the proof). However, $\theta = O(\tau)$ as $\tau \to \infty$ by the same ergodic theorem.

Indeed, in order to prove that $\theta = O(\tau)$ it suffices to consider integer values of τ. In this case,

$$\theta(\tau) = \sum_{k=0}^{\tau-1} T^k\eta; \quad \eta = \int_0^1 b(s)^2 d\langle M, M\rangle(s).$$

Here, η is an integrable random quantity, and T is the measure preserving map of the unit time shift. (The measure $d\langle M, M\rangle$ is stationary since the martingale M has stationary increments.) The ergodic theorem says, that $\theta(\tau)/\tau$ converges with unit probability to an integrable random quantity, in particular, $\theta(\tau) = O(\tau)$. Therefore, the last integral in (3.8) can also be neglected compared to $\varepsilon^2 f\tau$ as $\tau \to \infty$. Therefore $\log V(\tau)$ tends (linearly) to $-\infty$, and $V(\tau)$ tends to zero exponentially rapidly if ε is sufficiently small. □

4. Hamiltonian systems

We will now consider the stability (or rather instability) of a frictionless pendulum. In other words, we put $\delta = 0$ in (1.1). In what follows the presence of the small parameter ε is unimportant, so we put $\varepsilon = 1$ and arrive at the system

$$x' = y, \quad y' = (z'' + k^2)x. \tag{4.1}$$

We take process (2.1) for z. Note that this system does not depend on how we regard it: as a Stratonovich equation or as an Ito equation. The standard proof of Liouville's theorem transfers without change for the Stratonovich stochastic Hamiltonian equations. Thus, the Liouville measure is preserved under phase flow regardless of in what sense we understand the system of differential equations. We will consider it in Ito's sense. Instability of system (4.1) follows from the results in Ref. 13. Below we present much simpler direct arguments.

Consider the stochastic differential of the energy

$$H = \frac{1}{2}y^2 - \frac{1}{2}k^2x^2.$$

We have

$$dH = z''xy + \frac{1}{2}x^2dt = -(z+z')xydt + xydw + \frac{1}{2}x^2dt.$$

Hence it follows that (all the integrals in the proof here and henceforth are taken over the time interval $[0, T]$)

$$\mathbf{E}H(T) = H(0) - \mathbf{E}\int(z_t + z_t')x_ty_tdt + \mathbf{E}\int\frac{1}{2}x_t^2dt. \tag{4.2}$$

Suppose system (4.1) is stable with unit probability. Then, we have a fundamental system of neighborhoods U of zero which are stable under the phase flow. Indeed, if U' is a neighbourhood of zero which remains inside some other small neighborhood of zero under phase flow ϕ_t, we can put $U = \cup_{t\geq 0}\,\phi_t\,(U')$. Now we integrate (4.2) over an invariant neighborhood with respect to the Liouville measure. The result takes the form

$$\int_U \mathbf{E}H(T)d\lambda(x_0, y_0) = \int_U H(0)d\lambda(x_0, y_0) -$$

$$-\mathbf{E}\int(z_t + z_t')\int_U x_ty_td\lambda(x_0, y_0)dt + \mathbf{E}\int\int_U\frac{1}{2}x_t^2d\lambda(x_0, y_0)dt.$$

However, the inner integrals are independent of time t. Moreover, they are *deterministic* constants, since they are obviously so at the initial instant $t = 0$. Hence we obtain

$$\int_U \mathbf{E}H(T)d\lambda(x_0, y_0) = C_1 + C_2\mathbf{E}\int(z_t + z_t')dt + C_3T, \tag{4.3}$$

where C_3 is a *positive* constant. But $\mathbf{E}\int_0^T(z_t + z_t')dt = 0$, and we obtain that the integral on the left-hand side of (4.3) tends to infinity as $T \to \infty$. This contradicts the stability of the system (4.1).

We can avoid using the mathematical expectation $\mathbf{E}$ and work with sample paths. Namely, we can replace (4.2) with

$$H(T) = H(0) - \int(z_t + z_t')x_ty_tdt + \int\frac{1}{2}x_t^2dt + \int x_ty_tdw.$$

The same arguments lead to the equality

$$\int_U H(T)d\lambda(x_0, y_0) = C_1 + C_2\int(z_t + z_t')dt + C_3T + C_4w(T)$$

(In fact, $C_2 = -C_4$.) However it is clear that all terms on the right-hand side, apart from C_3T, will be $o(T)$. Thus, the left-hand side tends to $+\infty$ as $T \to \infty$. This again contradicts the stability of system (4.1).

Note, that, in general, the Hamiltonian flow corresponding to a Hamiltonian of the form $H + \xi U$, where $H = H(p, q)$ and $U = U(p, q)$ are smooth functions and ξ is white noise, *does not* preserve the symplectic form $\sum_i dp_i \wedge dq_i$ and the Liouville measure, if the Hamiltonian equations

$$dp = -\frac{\partial H}{\partial q}dt - \frac{\partial U}{\partial q}dw, \quad dq = \frac{\partial H}{\partial p}dt + \frac{\partial U}{\partial p}dw \tag{4.4}$$

are treated in Ito's sense. The Liouville measure is, however, preserved when the following condition is satisfied

$$(\mathrm{Hess}\,U)^* J \,\mathrm{Hess}\,U = 0; \quad J = \begin{Vmatrix} 0 & E \\ -E & 0 \end{Vmatrix}, \tag{4.5}$$

where Hess(U) means the Hessain (a matrix of second partial derivatives) of the function U and the asterisk denotes transposition.

These remarks enable us to formulate the following general result on the stochastic instability of Hamiltonian systems. Its proof is completely similar to the preceding arguments.

Theorem 2. *Let H be a strictly convex Hamiltonian such that the origin of coordinates* (0, 0) *is a stable equilibrium point for the corresponding Hamiltonian system. Consider a perturbed Hamiltonian of the form* $H + \xi U$, *where U is a smooth function which satisfies* (4.5), *non-plane at the origin and such that* $dU(0, 0) = 0$, *and* $\xi = dw/dt$ *is white noise. Then the origin of coordinates is unstable for the perturbed system.*

Proof. Applying Ito's formula we obtain

$$dH = \{U, H\}dw + \frac{1}{2}\left(\frac{\partial^2 H}{\partial^2(p, q)}h_U, h_U\right)dt,$$

where $\partial^2 H/\partial^2(p, q)$ is the Hessian of the function H, and $h_U = (\partial U/\partial p, -\partial U/\partial q)$ is the Hamiltonian vector field corresponding to the function U. Our assumptions guarantee that the second term in the preceding equality is everywhere non-negative, and is strictly positive in a small neighbourhood Σ of the origin, with some "thin" subset deleted. Here "thin" means, in particular, that the Liouville measure λ of this subset of the said neighbourhood, where

$$((\partial^2 H/\partial^2(p, q))h_U, h_U) \le \delta$$

tends to zero as $\delta \to 0$.

Now assume that the origin is *stable* for the perturbed system. This implies, that a small neighbourhood Ω of the origin exists such that no phase trajectory, with origin in Ω ever leaves this neighbourhood Σ. Now, we choose δ to be so small that the doubled measure of the set

$$\{(q, p) \in \Sigma; ((\partial^2 H/\partial^2(p, q))h_U, h_U) \le \delta\}$$

is less than the Liouville measure $\int_\Omega dpdq$ of the set Ω. Using Liouville's theorem we conclude that, for any t,

$$\int_\Omega \left(\frac{\partial^2 H}{\partial^2(p, q)}h_U, h_U\right)(q_t, p_t)d\lambda(q_0, p_0) \ge \frac{1}{2}\delta\,\mathrm{mes}(\Omega).$$

Hence it follows that

$$\mathbf{E}\int_\Omega H(q_t, p_t)d\lambda(q_0, p_0) \ge \frac{1}{2}t\delta\,\mathrm{mes}(\Omega),$$

which clearly contradicts our assumption that $(q_t, p_t) \in \Sigma$ if the time t is sufficiently long. □

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